Neyman-Pearson Lemma

How to find a MP (most powerful) test among all available tests for a particular H_0 against H_1? Neyman-Pearson lemma answers this.

For a simple null hypothesis, $H_0: \theta = \theta_0$ against a simple alternative $H_1: \theta = \theta_1$, this lemma has been proposed.

Statement. Let $K > 0$, be a constant and W be a critical region of size α such that

$$W = \left\{ \underline{x} \in \Omega : \frac{f(\underline{x}, \theta_1)}{f(\underline{x}, \theta_0)} > K \right\} \text{ —— ①}$$

Or $$W = \left\{ \underline{x} \in \Omega : \frac{L_1}{L_0} > K \right\}, A = \left\{ \underline{x} \in \Omega : \frac{L_1}{L_0} < K \right\} \text{ —— ②}$$

where L_0 and L_1 are the likelihood functions of the sample observations $\underline{x} = (x_1, x_2, \cdots, x_n)$ under H_0 and H_1 respectively. Then W is the most powerful critical region of the test hypothesis $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$.

Proof : Given, $P(\underline{x} \in W / H_0) = \int_W f(\underline{x} / H_0) d\underline{x} = \alpha$, we've to prove

In order to establish the lemma, we've to prove that there exists no other critical region, of size less than or equal to α, which is more powerful than ~~than~~ or equal to α, which is more powerful than critical region for W.

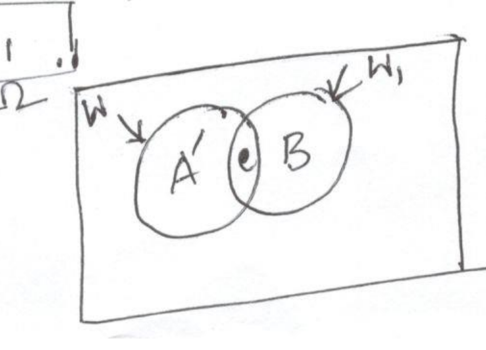
Let W_1 be another ~~the~~ size α_1 critical region for the same test. So, $P(\underline{x} \in W_1 / H_0) = \int_{W_1} L_0 \, d\underline{x} = \alpha_1 \leq \alpha$

For the critical region W, W,
power $= P(\underline{x} \in W / H_1) = 1 - \beta = \int_W L_1 \, d\underline{x}$.

Similarly, for the critical region W_1,
power $= P(\underline{x} \in W_1 / H_1) = 1 - \beta, = \int_{W_1} L_1 \, d\underline{x}$

$\boxed{\text{We'll prove } 1 - \beta > 1 - \beta_1 .}$

W and W_1 are overlapping
Let $W = A' \cup C$
and $W_1 = B \cup C$



Please note that C may be empty.

consider $\alpha_1 \leq \alpha$, then we have

$$\int_{W_1} L_0 \, d\underline{x} \leq \int_{W} L_0 \, d\underline{x}$$

$$\Rightarrow \int_{B \cup C} L_0 \, d\underline{x} \leq \int_{A' \cup C} L_0 \, d\underline{x}$$

$$\Rightarrow \int_{B} L_0 \, d\underline{x} \leq \int_{A'} L_0 \, d\underline{x}$$

So, $\boxed{\int_{A'} L_0 \, d\underline{x} \geq \int_{B} L_0 \, d\underline{x}} \longrightarrow$ ③

Consider ①, on W, $\frac{L_1}{L_0} > K$

i.e on W, $L_1 > KL_0$.

$$\Rightarrow \int_{W} L_1 \, d\underline{x} > K \int_{W} L_0 \, d\underline{x}.$$

$$\Rightarrow \int_{A'} L_1 \, d\underline{x} > K \int_{A'} L_0 \, d\underline{x} \geq K \int_{B} L_0 \, d\underline{x} \quad \text{[from ③]}$$

$$\Rightarrow \int_{A'} L_1 \, d\underline{x} \geq K \int_{B} L_0 \, d\underline{x}.$$

$$\Rightarrow \boxed{K \int_{B} L_0 \, d\underline{x} \leq \int_{A'} L_1 \, d\underline{x}.} \longrightarrow ④$$

consider, ② $\frac{L_1}{L_0} < K$ on A (Acceptance region, $A \cup W = \Omega$)

On A, $L_1 < KL_0$

on B, $L_1 < KL_0$ as $B \in A$ [complementation of W]

Thus, $\int_{B} L_1 \, dx < K \int_{B} L_0 \, d\underline{x} \leq \int_{A'} L_1 \, d\underline{x}$ [From ④]

$$\text{So, } \int_{B} L_1 \, d\underline{x} \leq \int_{A'} L_1 \, d\underline{x}.$$

$$\Rightarrow \int_{B \cup C} L_1 \, d\underline{x} \leq \int_{A' \cup C} L_1 \, d\underline{x}$$

$$\Rightarrow \int_{W_1} L_1 \, dx \leq \int_{W} L_1 \, dx.$$

$$\Rightarrow 1 - \beta_1 \leq 1 - \beta.$$

$\Rightarrow$ power of $W_1 \leq$ power of W.

$\therefore$ W is the best critical region.

Remark ① So Neyman-Pearson lemma gives the structure of the critical region of MP test.

In case of monotonic function, the critical region can be converted in terms of lower function as well. For example, if the critical region comes in terms of $e^x > \underset{\text{(constant)}}{c}$, we convert it in $x > c'$ since e^x is ↑ (monotonically increasing function) of x.

② Given the level of significance α, we can find the constant in critical region of MP test. Also given the critical region, one can find out the prob. of type I error or α.

③ The N-P lemma does not require the sample observations to be identically and independently distributed. The distributions specified under H_0 and H_1 need not even belong to the same family. Only thing is: both of the distributions under H_0 and H_1 should be completely specified.

Discussion on example

Get back to Example 2 where $X \geq 1$ (critical region) $H_0: \theta = 2$ against $H_1: \theta = 1$ for $f(x;\theta) = \theta e^{-\theta x}$, $0 \leq x < \infty$ (based on single obs.)

We already did α and β.

Question is $W: X \geq 1$ a MP critical region?

Try to solve the question by N-P lemma.

MP critical region will be of the form $\frac{f_{H_1}(x)}{f_{H_0}(x)} > k$.

$$f_{H_1}(x) = 1 \cdot e^{-x}; \quad f_{H_0}(x) = 2e^{-2x}; \quad x \geq 0$$

$$\therefore \frac{f_{H_1}(x)}{f_{H_0}(x)} = \frac{e^{-x}}{2e^{-2x}} > k$$

$$\Rightarrow e^x > 2k \longrightarrow \text{form of critical region}$$

e^x monotonic function of X; we convert the above form in $\boxed{X > k'}$ [k' constant].

So, given the value of α, you can figure out k', by $P_{H_0}[X > k'] = \alpha$. So $X \geq 1$ is a MP critical region or if we fix $\alpha = .05$, $K' = 1.49$ as

$P_{H_0}[X > K'] = .05$.

$$\Rightarrow \int_{K'}^{\infty} 2e^{-2x} = .05$$

$$\Rightarrow 2.\frac{e^{-2x}}{-2}\Big|_{K'}^{\infty} = .05 \quad \Rightarrow K' = 1.49.$$

so we reject Ho if $x > 1.49$
accept Ho if $x < 1.49$.

Example 4 On the basis of a single observation x from the following p.d.f. $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $x>0$, $\theta>0$ the null hypothesis $H_0: \theta = 1$ against the alternative hypothesis $H_1: \theta = 4$ is tested by using a set
$C = \{x; x > 3\}$ as the critical region.
Prove that the critical region C provides a most powerful test of its size. What is the power of the test?

Ans First Apply N-P lemma for finding out the MP critical region.

$$\frac{f(x/H_1)}{f(x/H_0)} = \frac{\frac{1}{4}e^{-x/4}}{\frac{1}{1}e^{-x/1}} > K.$$

$$\Rightarrow \quad \frac{1}{4}e^{+\frac{3}{4}x} > K.$$

$$\Rightarrow \quad e^{3/4x} > 4K$$

$$\Rightarrow \quad x > K'$$

$\uparrow$
MP critical region format.

[As $e^{\frac{3}{4}x} \uparrow x$ convert the critical region into lower function x]
or you may use both sides logarithmic transformation

$\therefore$ C is most powerful critical region.
$\{x > 3\}$

Now you might find out the size by using $P[X > 3/H_0]$

$$\text{Power} = P[X > 3/H_1]$$

$$= \int_{3}^{\infty} \frac{1}{4}e^{-x/4}dx.$$

$$= \frac{1}{4}.4e^{-x/4}\Big|_{3}^{\infty}$$

$$= e^{-3/4}$$

Example 5 [Special type (Why ?)]

It is required to test Ho against H1 from a single observatio x, where Ho is the $f(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $-\infty<x<\infty$.

and H1 is $f(x)=\frac{2}{\Gamma\frac{1}{4}}e^{-x^4/2}$, $-\infty<x<\infty$.

obtain most powerful test with level of significance α in this case.

Answer

$$\frac{f_{H1}(x)}{f_{Ho}(x)} = \frac{\frac{2}{\Gamma\frac{1}{4}}e^{-x^4/2}}{\frac{1}{\sqrt{2\pi}}e^{-x^2/2}} = \underbrace{\boxed{\frac{2\sqrt{2}\,\Gamma\frac{1}{2}}{\Gamma\frac{1}{4}}}}_{\text{constant}} e^{x^2/2} > K.$$

$\therefore\ e^{x^2/2} > K'$ (const)

$\Rightarrow x^2 > K''$ (const) $\left[e^{x^2/2}\uparrow x^2\right]$ $\quad x > \sqrt{K''}$ or $x<-\sqrt{K''}$

Most powerful critical region

We reject Ho if $X>\sqrt{K''}$ or $X<-\sqrt{K''}$

How to find the constant K''?

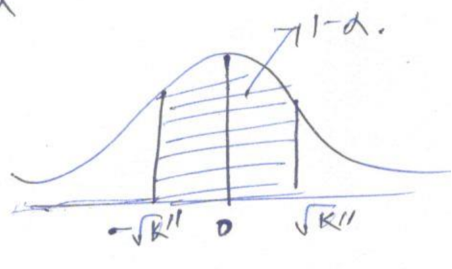
Use size condition

$$P[X \in W/H_0] = \alpha$$

$$\Rightarrow P_{H_0}[X>\sqrt{K''} \cup X<-\sqrt{K''}] = \alpha$$

$$\Rightarrow P_{H_0}[-\sqrt{K''}<X<\sqrt{K''}] = 1-\alpha$$

$$\Rightarrow \int_{-\sqrt{K''}}^{\sqrt{K''}} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = 1-\alpha$$



$$\Rightarrow 2\Phi(\sqrt{K''})-1 = 1-\alpha$$

$$\Rightarrow \alpha = 2-2\Phi(\sqrt{K''})$$

$$\Rightarrow 2\Phi(\sqrt{K''}) = 2-\alpha$$

$$\Rightarrow \Phi(\sqrt{K''}) = 1-\alpha/2$$

$$\Rightarrow \sqrt{K''} = \Phi^{-1}(1-\alpha/2)$$

$\Phi(\sqrt{K''})-(1-\Phi(\sqrt{K''})$

2Φ

You've to look to Normal table for any numeric value of α.

Example 6

Let $x_1, x_2, \dots, x_n$ be an random sample from discrete distribution with probability mass function.

Under H_0 : $f(x) = \frac{e^{-1}}{x!}$; $x = 0, 1, 2, \dots$

Under H_1 : $f(x) = \frac{1}{2^{x+1}}$; $x = 0, 1, 2, \dots$

Obtain the critical region of Most powerful test of level α. Also find the power of the test for the case $n=1$ and $K=1$.

Answer Construct the joint probability mass functio as number of observations more than 1.

By N-P lemma the critical region $$\frac{f_{H_1}(\underline{x})}{f_{H_0}(\underline{x})} = \frac{1}{(2)^n \cdot 2^{\sum_1^n x_i}} \cdot \frac{\prod_{i=1}^{n} x_i!}{e^{-n}} > K$$

constant $$\Rightarrow \left(\frac{e}{2}\right)^n \frac{\prod_{i=1}^{n} x_i!}{2^{\sum_1^n x_i}} > K.$$

$$\Rightarrow \frac{\prod x_i!}{2^{\sum x_i}} > K'$$

$$\Rightarrow -\sum x_i \log 2 + \sum \log x_i > k'$$

$$\Rightarrow -\cdot 693 \sum x_i + \sum \log x_i > k'$$

$$\Rightarrow -\sum x_i + 1.443 \sum \log x_i > k''$$

MP critical region $\sum x_i < k''$

We reject H_0 for a sample $(x_1, x_2, \dots, x_n)$ if
$$-\sum x_i + 1.443 \sum \log x_i > k''$$

K'' can be determined $P_{H_0}\left(-\sum x_i + 1.443 \sum \log x_i > k'' / H_0\right) = \alpha$

The above expression dominated by $\sum x_i$.

Now, when $K=1$, $n=1$] $P_{H_0}(\sum x_i < k''/H_0) = \alpha$. when $x = 0, 1, 2, \dots$

$$P_{H_0}(-x + 1.443 \log x > 1 / H_0) = \alpha$$

~~power $P_{H_1}(-x + 1.443 \log x > 1 / H_1)$~~

$$P_{H_1}(x < 1 / H_1)$$

$$= P_{H_1}(x = 0 / H_1) = \frac{1}{2}$$